Problem 1. (7 pts) The residents of a planet far, far away that has no natural moons decide to build a moon-sized battle station for defensive purposes. The planet has mass $M$, while the battle station (labeled '1') has mass $m$ and travels in a circular orbit with radius $r$ and period $T_0$. Then someone suggests that two battle stations are better than one, so the residents plan for a second, identical battle station ('2') to travel in a circular orbit with the same radius $r$, but exactly opposite the first battle station. Is the orbital period for the two-station case larger than, smaller than, or the same as $T_0$? Explain.

For circular orbits: \[ F_g = m \frac{v_i^2}{r} = m \omega_i^2 r \]

Where $\omega_i$ = angular velocity \[ T = \frac{2\pi}{\omega} = \text{period} \]

Gravitational Force on orbiting body

**One-station case** \[ F_g = F_p = m \omega_0^2 r = m \left( \frac{2\pi}{T_0} \right)^2 r \]

From planet \[ S_0 \quad T_0 = \left[ \frac{4\pi^2 m r}{F_p} \right]^{\frac{1}{2}} \quad \text{Period w/one-station} \]

**Two-station case** \[ F_g = F_p + F_s = m \omega_0^2 r = m \left( \frac{2\pi}{T_b} \right)^2 r \Rightarrow T_e = \left[ \frac{4\pi^2 m r}{F_p + F_s} \right]^{\frac{1}{2}} < T_0 \]

Period w/two stations

Orbital period for two-station case is smaller

2 pts
Problem 2. (7 pts) Air flows through a variable-area channel that has a rectangular cross-section. The channel is surrounded by air at atmospheric pressure. At the inlet to the channel on the left side, the channel has height $d_1$ and width $w$ in the out-of-page direction, the air moves at velocity $V_1$, and the pressure in the flowing air is equal to atmospheric pressure. Then the channel tightens down to height $d_2$, while the width stays at $w$. In this narrower section there's a rigid massless door that's hinged so that it can swing both up and down. If the air can be treated as an ideal fluid, and you want to keep the door in the position shown, do you have to pull up or push down on the door? Explain mathematically. (You can ignore gravity.)

\[ V_1 \]

\[ \uparrow \quad d_1 \quad \uparrow \quad d_2 \quad \uparrow \]

* **Mass Conservation:** \[ V_1 A_1 = V_2 A_2 \], so \[ \frac{V_2}{d_2} - \frac{V_1}{d_1} > V_1 \] (1 pt)

* **Bernoulli Eq.:**
  \[ P_1 + \frac{1}{2} \rho V_1^2 = P_2 + \frac{1}{2} \rho V_2^2 \] (ignoring gravity) (2 pts)

So \[ \frac{P_2 - P_1}{\rho} = \frac{1}{2} \rho (V_1^2 - V_2^2) < 0 \] since \[ V_2 > V_1 \] (2 pts)

So \[ P_2 < P_{atm} \], means pressure smaller inside the channel @ door than outside (2 pts)

\[ \copyright \quad \text{You need to pull up on the door to keep it in place.} \] (2 pts)
**Problem 3.** (12 pts) A very heavy, perfectly rigid beam, with mass \( M = 1.00 \times 10^5 \) kg, sits, centered, across two concrete, vertical legs that are a distance \( L = 20 \) m apart. The legs are square cylinders with side length \( w = 0.300 \) m. When no load is applied to either leg, one leg has length \( 2H = 8.00 \) m and the other, which sits on a step with height \( H = 4.00 \) m, has length \( H \). When the beam alone is placed on the legs, the legs may compress different amounts. To compensate for this you can fasten a block, with mass \( m = 2M \), anywhere along the length of the beam. What distance \( l \) from the left leg should you put this block so the beam winds up horizontal?

\[
\text{Young's modulus: } E_i = \frac{E}{A_i} \quad \text{Applied stress on leg } i \quad \text{Strain on leg } i
\]

- **Know:** \( \frac{\Delta L}{L} \) same change in absolute length for both legs.
- **Want:** \( \frac{\Delta L_L}{L_L} = \frac{\Delta L_R}{L_R} \) since \( L_L = 2H = 2L_R \), we need \( \frac{\Delta L_L}{L_L} = \frac{1}{2} \frac{\Delta L_R}{L_R} \).
- Also \( A \) for both legs is same, as is \( E \) (same material).

So we need

\[
F_L = A_L E_L \frac{\Delta L_L}{L_L} = A_R E_R \frac{1}{2} \frac{\Delta L_R}{L_R} = \frac{F_R}{2} = F_L
\]

- Consider equilibrium.

\[
\begin{align*}
F_R & = 2M \\
F_M & = \text{plug in } F_R = 2M \\
F_L & = \text{plug in } F_R = 2M \\
\end{align*}
\]

**Forces:** \( F_L + F_R = F_M + F_m = Mg + mg = 3Mg = \frac{3}{2} F_R \) (using \( \theta \)) \( \Rightarrow F_R = 2Mg \)

**Torques:** \( F_m \frac{L}{2} + F_m l - F_R L = 0 \) (pivot point @ left leg)

\( \Rightarrow Mg \frac{L}{2} + 2MgL - F_R L = 0 \) (plug in \( F_R = 2Mg \))

\( \Rightarrow Mg \frac{L}{2} - 2MgL + 2MgL = 0 \) cancel \( Mg \) terms

\( \Rightarrow 2MgL = 2MgL \) cancel \( Ml \) terms

So \( l = \frac{1}{2} \left( 2L - \frac{L}{2} \right) = \frac{3L}{4} = \frac{15}{4} \) m

Put block this far from left leg.
Problem 4. (Total 12 pts) An enormous cylindrical ‘cup’, open at the top and with diameter $D = 25$ cm, is filled with water ($\rho = 998$ kg/m$^3$) to a height $H$. You would like to drink from the cup using a straw with length $L = 60$ cm and diameter $d = 6$ mm. Assume that the straw is vertical and reaches nearly to the bottom of the cup (leaving just enough space for water to get into the straw), and that the pressure at the top of the straw is the pressure in your mouth.

(a) (8 pts) Suppose $H = 32$ cm and the straw is filled with water. What gauge pressure (in Pascals) do you need to generate in your mouth in order to get the water to move upward in the straw? Find this pressure by considering a force balance on the water in the straw.

(b) (4 pts) If you can generate a gauge pressure of $-4.0 \times 10^3$ Pa in your mouth, and the cup starts with $H = 32$ cm, what’s the height $H$ of water left in the cup after you’ve sucked out all that you can?

\begin{align*}
\text{To get water to move upward, upward force must be }> \text{ downward force} \quad 2 \text{ pts}
\end{align*}

\begin{align*}
\text{So } \quad (P_{\text{atm}} + \rho g H)A > (\rho g L + P_3)A \\
\Rightarrow \quad \frac{P_3 - P_{\text{atm}}}{\rho g} < \frac{H - L}{A} \quad \text{(gauge in mouth)} \quad \text{(1 pt)}
\end{align*}

\begin{align*}
\rho \quad P_3 - P_{\text{atm}} < 998 \frac{\text{kg}}{\text{m}^3} \left(0.32 \text{ m} - 0.60 \text{ m}\right) = -2.74 \times 10^3 \text{ Pa} \quad \text{(2 pts)}
\end{align*}

\begin{align*}
\text{Gauge P in mouth must be less than this} \quad \text{(1 pt)}
\end{align*}

\begin{align*}
\text{What's } H \text{ when upward/downward forces balance?} \quad \text{(For this } P_3 - P_{\text{atm}}\text{)}
\end{align*}

\begin{align*}
\text{Using } \star \text{ above } \Rightarrow \quad P_3 - P_{\text{atm}} = -4.0 \times 10^3 \text{ Pa} = \rho g (H - L) \quad \text{(2 pts)}
\end{align*}

\begin{align*}
\Rightarrow \quad H = \frac{-4.0 \times 10^3 \text{ Pa}}{998 \frac{\text{kg}}{\text{m}^3} \cdot 9.8 \frac{\text{m}}{\text{s}^2}} + 0.6 \text{ m} = 0.19 \text{ m} \quad \text{(2 pts)}
\end{align*}

\begin{align*}
\text{Level after which you can't suck out any more} \quad \text{(2 pts)}
\end{align*}
Problem 5. (Total 12 pts) Two bodies, where  $m_1 = 2.0 \times 10^{20}$ kg and $m_2 = 8.0 \times 10^{20}$ kg, sit fixed in outer space a distance $D = 7.5 \times 10^7$ m apart. Consider a space probe, with mass $M = 1100$ kg, that interacts with the two bodies.

(a) (6 pts) If the probe is on the line connecting the two bodies, at what distance $d$ from $m_2$ will there be no net gravitational force on the probe?

(b) (6 pts) Now assume that the probe is at the position identified in part (a). If it is suddenly given some velocity $v_0$ in the $x$-direction, will it continue to go straight, turn right (toward $y < d$), or turn left (toward $y > d$) in the early part of its travel? Explain briefly.

[Diagram and equations related to gravitational forces and motion are depicted here.]

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(a) Gravitational force from $m_1$: $F_1 = \frac{G m_1 M}{(D-d)^2}$; from $m_2$: $F_2 = \frac{G m_2 M}{d^2}$ (1 pt)

Set equal: $\frac{m_1 m}{(D-d)^2} = \frac{m_2 m}{d^2} \Rightarrow \frac{D-d}{\sqrt{m_1}} = \frac{d}{\sqrt{m_2}} \Rightarrow D-d = d \sqrt{\frac{m_1}{m_2}}.$ (2 pts)

So $d = \frac{D}{1 + \frac{m_1}{m_2}} = \frac{7.5 \times 10^7}{1 + \frac{1}{2}} = 5.0 \times 10^7$ m (2 pts)

(b) Consider: $\theta_1$, $\theta_2$, $\Delta x$, i.e., mass $M$ has moved $\Delta x$. Need to know net force in $y$-direction (1 pt)

For $m_1$: $F_{1, y} = F_1 \cos \theta_1 = \frac{m_1 m}{\sqrt{\gamma_1^2}} \frac{d_2}{\gamma_1} = \frac{m_1 M d}{2(\frac{d_2}{\gamma_1} \Delta x^2)^{3/2}}$ (1 pt)

For $m_2$: $F_{2, y} = F_2 \cos \theta_2 = \frac{m_2 M d}{\sqrt{\gamma_2^2}} \frac{d}{\gamma_2} = \frac{m_2 M d}{(d^2 + 4 \Delta x^2)^{3/2}}$ (1 pt)

Now compare to (a): Note $m_1 = \frac{m_2}{4}$, so $F_{1, y} = \frac{m_2 M d}{8(\frac{d_2}{4} \Delta x^2)^{3/2}} = \frac{M M_2 d}{(d^2 + 4 \Delta x^2)^{3/2}}$ (1 pt)

Since $\theta = 4^{3/2}$, $F_{2, y} > F_{1, y}$, probe will turn right (2 pts)