Problem 1 - Short answer. (5 pts)

In class we defined the Froude number, which relates inertial flow effects to buoyancy effects, as \( Fr = \frac{V}{\sqrt{gL}} \), where \( V \) is a characteristic flow velocity, \( g \) is gravity, and \( L \) is some characteristic flow length scale. (It’s pretty easy to see where the Froude number comes from by inspection of the Bernoulli equation, for example.) What would be a reasonable dimensionless number to relate pressure effects to buoyancy effects? Don’t use the full Pi theorem procedure.

- From the hint: look at the Bernoulli eq. \( \frac{P}{\rho} + \frac{V^2}{2} + gzh = \text{const.} \)

- Term I: Pressure; Term II: Inertial effects; Term III: Buoyancy/Gravity

- Note: The Froude number is proportional to \( \sqrt{\frac{\text{III}}{\text{III}}} \rightarrow \text{with} \ L \ \text{in place of} \ z \)

- Similarly, a dimensionless # to relate pressure and buoyancy can be formed as \( \frac{\text{II}}{\text{III}} \), i.e.

\[
\frac{P}{\rho g L}
\]

(2 pts reasoning (no credit for Pi theorem))

(3 pts result)
Problem 2. (Total 25 pts) A tall dam holds back a humongous reservoir of water, at 20°C, that is to be used for power generation. Consider the two flow configurations shown. In the left configuration, water flows through the power-generating turbine via a horizontal penstock located a depth $H$ below the reservoir's surface, and discharges to the atmosphere. In the right configuration, the penstock inlet is at the surface of the reservoir, but the outlet is again at $H$ below the surface. You can think of a turbine as the opposite of a pump; it converts mechanical energy from the flow into work (assume here that this conversion is 100% efficient).

Throughout this problem $H = 200$ m, the penstocks are galvanized iron with diameter $D = 1.2$ m, and the penstock outlet(s) are at a horizontal distance $L = 180$ m from the inlet.

a) (5 pts) If the water could be treated as inviscid and both turbines were disconnected, which of the two configurations would yield a higher volume flow rate, $Q$, or would they be the same? Explain your answer mathematically.

b) (10 pts) Consider the configuration on the left, but pretend the turbine isn’t there. Treating the water as viscous, and if the inlet to the penstock is square-edged, what’s $Q$?

c) (10 pts) Consider again the configuration described in b), but now with the turbine in place and operational. If the turbine generates $2.6 \cdot 10^7$ Watts of power, what’s $Q$? Ignore any minor losses associated with the turbine and assume that the pipe length is unchanged.

**a** With the given assumptions a) can apply Bernoulli Eq. Between free surface and Pipe outlet. Get
\[
\frac{P_a}{\rho} + \frac{V_a^2}{2} + g\,Z_a = \frac{P_b}{\rho} + \frac{V_b^2}{2} + g\,Z_b
\]
where $P_a = P_b$
\[
z_a - z_b = H
\]
\[
V_a = 0
\]

So $V_b = \sqrt{2gH}$ same for both configurations

Q = V_b \cdot A would be the same for both configurations.

**b** Write energy Eq. From free surface to outlet
\[
\left[ \frac{P_a}{\rho} + \frac{V_a^2}{2} + g\,Z_a \right] - \left[ \frac{P_b}{\rho} + \frac{V_b^2}{2} + g\,Z_b \right] = h_l + h_{lm} = \frac{1}{2} \sqrt{\frac{V_b^2}{2}} + \frac{K\,V_b^2}{2}
\]

Free surface ($V_a = 0$) Outlet (Assume $V_b = 1$) Major losses Minor losses (inlet/penstock) (Table 8.2)
(Extra workspace for problem 2)

\[ \text{REARRANGE } \begin{align*}
\frac{H}{g} &= g \left( z_a - z_B \right) - \frac{V_b^2}{2} = \frac{V_b^2}{2} \left( \frac{f}{D} + 0.5 \right) \\
\Rightarrow \frac{V_b^2}{2} \left( \frac{f}{D} + \frac{1}{2} + \frac{1}{4} \right) &= gH
\end{align*} \]

For \( f \): \( \frac{e}{D} = \frac{0.15 \times 10^{-3}}{1.2m} = 0.00125 \)  

Try asymptotic \( f = 0.124 \) (1 pt)

\[ \overline{V_b}^2 = gH \left( 15\text{ft} + 0.75 \right)^{-1} \Rightarrow \overline{V_b} = \sqrt{\frac{1.3 \, \text{ft}^2 \cdot 200 \, \text{in}}{15 \times 0.0124 + 0.75}} \] 

\[ \overline{V_b} = 34.2 \, \text{m/s} \] (1 pt)

\[ \text{Check } Re = \frac{\overline{V_b} \cdot D}{\nu} = \frac{34.2 \, \text{m/s} \cdot 1.2m}{1.01 \times 10^{-6} \, \text{m}^2/\text{s}} = 4.1 \times 10^6 \quad \text{consistent with } f \text{ above} \] (1 pt)

So \( Q = \frac{\overline{V_b} \cdot \pi \left( \frac{D}{2} \right)^2}{\lambda_{\text{pipe}}} = 34.2 \, \text{m}^3/\text{s} \cdot 3.14 \cdot (0.6m)^2 = \frac{38.7 \, \text{m}^3}{s} \) (3 pts)

\[ \square \text{ WITH TURBINE IN PLACE: USE EQ. } \begin{align*}
\text{again, but with an extra } \Delta h \text{ term for } \Delta h_{\text{turb}} \text{ THE TURBINE} \\
\text{So RHS of } \begin{align*}
\overline{V_b}^2 = \frac{gH}{2} \left( \frac{f}{D} + 0.5 \right) + \Delta h_{\text{turb}} \quad \text{2 pts}
\end{align*} \\
\text{AND, REARRANGING } \Rightarrow gH = \overline{V_b}^2 \left( 15\text{ft} + 0.75 \right) + \Delta h_{\text{turb}} \rightarrow \text{ANALOGY OF TURBINE W/PUMP: } \\
\Delta h_{\text{turb}} = \frac{W}{\rho \overline{V_b} \cdot A_{\text{pipe}}} \\
\text{So } gH = \overline{V_b}^2 \left( 15\text{ft} + 0.75 \right) + \frac{W}{\rho \overline{V_b} \cdot A_{\text{pipe}}} \quad \text{2 pts}
\end{align*} \]

\[ \text{Plug in } \overline{V_b} \text{ in } \begin{align*}
1960 \frac{\text{m}^3}{s^2} = \overline{V_b}^2 \left( 1.68 \right) + \frac{2.6 \times 10^7 \text{ kg} \cdot \text{m}^2}{918 \text{ kg} \cdot \overline{V_b} \cdot \overline{T} (0.6 \text{m})^2} = 1.68 \overline{V_b}^2 + 2.31 \times 10^9 \frac{\text{m}^4/s^3}{\overline{V_b}^2} \\
\end{align*} \] (guess \( f = 0.024 \) as above)

\[ \text{Trial & Error: Find } \overline{V_b} = 25 \, \text{m/s} \text{ gives RHS of } \Rightarrow 1974 \frac{\text{m}^2}{s^2} \approx \text{LHS} \]

\[ \text{Check: } \text{Re} = \frac{25 \cdot 1.2}{1.01 \times 10^{-6}} = 3.16 \times 10^7 \text{ consistent w/f} \] (1 pt)

\[ \text{So } Q = \overline{V_b} \cdot \overline{T} (0.6 \text{m})^2 = \frac{25 \cdot 3 \, \text{m}^3}{s} \] (3 pts)
Problem 3. (Total 20 pts) Consider a large, open tank of water at 20°C. A siphon tube with diameter \( d = 0.6 \text{ cm} \) extends vertically from the tank, reaches a height \( H_1 = 2.1 \text{ m} \) above the water’s surface, turns around to point downward for a vertical distance \( H_2 = 0.8 \text{ m} \), then connects to the top of a large horizontal pipe through which air at 20°C flows steadily with volume flow rate \( Q \). The siphon tube is initially empty. At the point where the siphon tube joins the pipe, the pipe diameter is \( D_1 = 0.3 \text{ m} \). Later, the pipe expands to a diameter \( D_3 = 0.42 \text{ m} \), after which the air exits into the atmosphere. Treat the airflow in the pipe as inviscid, and ignore the effect of gravity on the air throughout.

![Diagram](image)

a) (3 pts) Explain how it is that water from the tank can end up in the airflow in the pipe.

b) (8 pts) What’s the minimum volume flow rate of air, \( Q_m \), for which water from the tank will end up in the airflow in the pipe?

c) (7 pts) Once the siphon tube is filled with water, if the flow rate of air is still \( Q_m \), what’s the volume flow rate of water, \( Q_w \), through the siphon into the airflow? Treat the water as inviscid.

d) (2 pts) How would the answer to c) change if the water were treated as viscous? Explain.

\[ a \] As the pipe airflow speeds up, its pressure drops. The pressure at the free surface level in the siphon tube is atmospheric, so as the \( P \) at the pipe end falls, water gets sucked upward, eventually reaching the top of the siphon tube and falling into the pipe.

\[ b \] To get water into pipe: need \( P_1 + \rho g H_1 = P_{\text{atm}} \) so \( P_1 \) is low enough to sustain hydrostatic \( P \) for a height \( H_1 \) of water.

In pipe: apply Bernoulli

\[
\frac{P_1}{\rho_{\text{air}}} + \frac{V_1^2}{2} + g z_1 = \frac{P_2}{\rho_{\text{air}}} + \frac{V_2^2}{2} + g z_2
\]

Use \( z_1 = z_2 \), \( V_1 D_1^2 = V_2 D_2^2 \) or \( Q_1 = Q_2 \) and \( V_1 = \frac{Q_1}{A_1} \).
(Extra workspace for problem 3)

\[
\frac{P_1 - P_{\text{atm}}}{P_{\text{air}}} = \frac{1}{2} (V_2^2 - V_1^2) = \frac{1}{2} \left( \frac{D_1^4}{D_2^4} - 1 \right) = \frac{Q_1^2}{2A_1^2} \left( \frac{D_1^4}{D_2^4} - 1 \right)
\]

\[
-\frac{\rho g \Delta h_1}{P_{\text{air}}} = \frac{Q_1^2}{2\pi^2 (D_2^4/2)^4} \left( \frac{D_1^4}{D_2^4} - 1 \right)
\]

2 pts

Plug in #’s:

\[
\frac{998 \text{ kg/m}^3 \cdot 9.8 \text{ m/s}^2 \cdot 2.1 \text{ m}}{1.21 \text{ kg/m}^3} = \frac{Q_1^2}{2\pi^2 (0.15 \text{ m})^4} \left( \frac{0.34}{0.42} \right)
\]

\[
-1.70 \cdot 10^{-4} \frac{\text{m}^2}{\text{s}^2} = Q_1^2 (-74.1 \cdot \frac{1}{\text{m}^2})
\]

2 pts

\[
Q_m = 15.1 \frac{\text{m}^3}{\text{s}}
\]

Minimum Q to get water into air flow

(Note: This gives \( V_1 = \frac{6.1 \cdot 10^{-3} \text{ m}^3}{\pi (0.15 \text{ m})^2} = 2 \cdot 10^{-2} \text{ m} \))

[III]

Look at siphon tube:

\( \uparrow \)

Filled w/water

1 pt

\([0] \Rightarrow P_1 = -\rho g H_1 + P_{\text{atm}} \) (For \( Q_m \) as above)

Apply Bernoulli from \([0] \Rightarrow P_0 = P_{\text{atm}}, V_0 = 0 \) to \([1]\)

\[
\frac{P_0}{P_{\text{H}2O}} + \frac{V_2^2}{2} + g \frac{Z_2}{g} = \frac{P_1}{P_{\text{H}2O}} + \frac{V_1^2}{2} + g \frac{Z_1}{g}
\]

Not same \( V_1 \) as in \([II]\)

2 pts

Rearrange:

\[
\frac{V_2^2}{2} = \frac{P_0 - P_1}{P_{\text{H}2O}} + g \left( Z_2 - Z_1 \right) \Rightarrow V_1 = \left[ 2g H_1 + 2g (H_2 - H_1) \right]^{1/2}
\]

2 pts

\[
= \left[ 2g H_2 \right]^{1/2} = 2.96 \frac{\text{m}}{\text{s}}
\]

2 pts

\[
Q_w = V_1 \cdot A_{\text{siphon}} = 3.96 \frac{\text{m}^3}{\text{s}} \cdot \pi \left( \frac{d = 6 \cdot 10^{-3} \text{ m}}{2} \right)^2 = 1.1 \cdot 10^{-4} \frac{\text{m}^3}{\text{s}} = Q_w
\]

2 pts

Differences in the water will slow down the flow for a given pressure difference

\( \Rightarrow \) \( Q_w \) will be smaller

2 pts