

530.327 - Introduction to Fluid Mechanics - Su
 Statics: Forces on a submerged, curved surface.

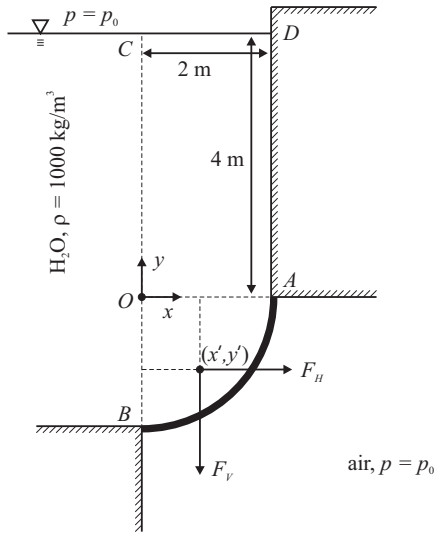


Figure 1: Hydrostatic forces on a submerged, curved surface.

We are interested in the problem depicted in Figure 1, involving a submerged, curved surface. What we want to know is, what is the hydrostatic force (magnitude, direction, and point of application) acting on the arc AB ?

The arc AB is circular, with radius 2 m. The fluid above the arc section is water; below AB is air at atmospheric pressure, p_0 , which is also the pressure on the free surface. The density of water is 1000 kg/m^3 , and the value of the gravitational acceleration is $g = 9.8 \text{ m/s}^2$. We can treat the problem as two-dimensional, i.e. the geometry is constant in the coordinate direction that runs perpendicular to the page. The forces that we are interested in are thus forces per unit length in the out-of-page (z) direction. In particular we'll consider a section of the geometry that spans one meter in the z -direction.

We will define the vertical component of the force as F_V and the horizontal component as F_H , and we will label the point through which the force acts as (x', y') , where the origin of the $x - y$ coordinate system is shown in the figure as the point O .

Vertical component. Let's look first at the vertical component of the force, F_V . We learned already that the magnitude of the of the vertical force component acting on a submerged surface is dependent on the volume of fluid that lies above that surface. (Strictly speaking, the atmospheric pressure above the free surface is involved too. But, because the space below the arc AB is occupied by air at atmospheric pressure, we don't need to worry about the pressure, p_0 , due to the air acting on the free surface of the water, since those forces balance.)

The relevant volume of fluid is that contained in the region $ABOCD$, which we can subdivide into the region $AOCDA$, which we'll call region 1, and the region $ABOA$ (region 2), as seen in Fig. 2. The weight of the water in region 1 (remembering that we're interested in volumes that span 1 meter in the z -direction) is

$$F_{V1} = \rho g (\text{Volume of region 1}) = 1000 \frac{\text{kg}}{\text{m}^3} \times 9.8 \frac{\text{m}}{\text{s}^2} \times 8 \text{ m}^3 = 78.4 \text{ kN},$$

while the weight of water in region 2 is

$$F_{V2} = 1000 \frac{\text{kg}}{\text{m}^3} \times 9.8 \frac{\text{m}}{\text{s}^2} \times \frac{\pi}{4} 4 \text{ m}^2 \times 1 \text{ m} = 30.8 \text{ kN},$$

so the total vertical force acting on the arc AB is

$$F_V = (78.4 + 30.8) \text{ kN} = 109.2 \text{ kN}. \quad (1)$$

Now we need to know the horizontal position, x' , through which the vertical force acts. Another thing we've learned is that the vertical force of a volume of fluid acts through the centroid of that volume. We

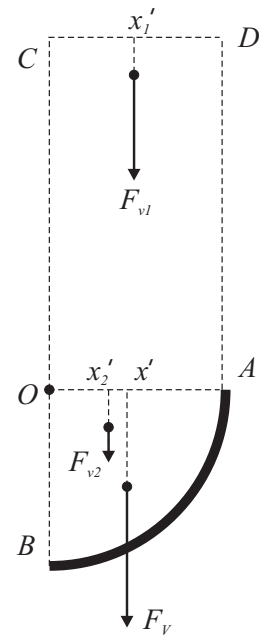


Figure 2: Vertical forces on the submerged, curved surface.

can use this information to look at the moments of the vertical forces around the line $x = 0$. Denote the horizontal coordinate of the centroid of region 1 as x'_1 , and that of centroid of region 2 as x'_2 . We know that moments can be added, so the moment of the total vertical force, F_V , acting through x' , is equal to the sum of the moment of F_{V1} acting through x'_1 , and the moment of F_{V2} acting through x'_2 . Region 1 is just a rectangle, so its horizontal centroid is $x'_1 = 1$ m. Meanwhile, the centroid of the quarter-circle $ABOA$ is known from basic calculus to be located at $x'_2 = 4r/3\pi = 0.85$ m. Equating moments, we then have

$$x'F_V = x' \times 109.2 \text{ kN} = x'_1F_{V1} + x'_2F_{V2} = 1 \text{ m} \times 78.4 \text{ kN} + 0.85 \text{ m} \times 30.8 \text{ kN} = 104.6 \text{ kN} \cdot \text{m},$$

giving us

$$x' = 0.96 \text{ m}. \quad (2)$$

Horizontal component. For the horizontal component of force, we make use of the result that the horizontal force (in the x -direction) on a curved, submerged surface is the same as the force that would act on a projection of that surface into the vertical plane that's perpendicular to the x -axis. Figure 3 depicts this situation for the arc AB . The line $A'B'$ is the vertical projection of

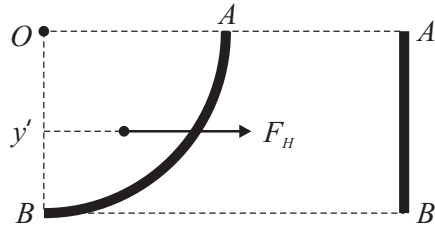


Figure 3: The projection, $A'B'$, of the curved surface, AB , used to determine the horizontal force component.

AB .

To determine the horizontal component of force acting on $A'B'$, we use the relation

$$F_H = p_c A,$$

where p_c is the pressure acting on the centroid of $A'B'$, and A is the area of the surface. Obviously, the centroid is located at a depth of $h_c = 5$ m (see Fig. 1), and the area of a meter-long section of $A'B'$ is 2 m^2 , so we can write

$$F_H = \rho g h_c A = 1000 \frac{\text{kg}}{\text{m}^3} \times 9.8 \frac{\text{m}}{\text{s}^2} \times 5 \text{ m} \times 2 \text{ m}^2 = 98.0 \text{ kN}. \quad (3)$$

(Note that we are ignoring the p_0 term that appears in the force, because that term is balanced by the force from the air that lies below the arc AB .) To find the line of action of the horizontal force on $A'B'$, we use the expression

$$y' = y_c - \frac{\rho g}{F_H} \int_A \hat{y}^2 dA, \quad (4)$$

where $y_c = -1$ m is the vertical coordinate of the centroid of $A'B'$, and $\hat{y} \equiv y - y_c$ is the vertical coordinate that's centered at y_c . The minus sign appears in Eq. 4, and $y_c < 0$, because y is defined here as being positive upward. For $A'B'$ the integral is easily evaluated as

$$\int_A \hat{y}^2 dA = \int_{-1 \text{ m}}^{1 \text{ m}} \hat{y}^2 d\hat{y} \cdot 1 \text{ m} = \frac{2 \text{ m}^4}{3},$$

remembering again that we're dealing with a meter-long section of $A'B'$. Plugging this into the equation for y' , we get

$$y' = -1 \text{ m} - \frac{1000 \frac{\text{kg}}{\text{m}^3} \times 9.8 \frac{\text{m}}{\text{s}^2} 2 \text{ m}^4}{98.0 \text{ kN}} \frac{2}{3} = -1.07 \text{ m}. \quad (5)$$

Result. To summarize the results of (1), (2), (3) and (5), the vertical component of force is 109.2 kN, acting downward through $x' = 0.96 \text{ m}$, and the horizontal component of force is 98.0 kN, acting in the positive x -direction through $y' = -1.07 \text{ m}$. Thus the net force acting on the arc AB is 146.7 kN, acting through $(x', y') = (0.96 \text{ m}, -1.07 \text{ m})$, at an angle 48.1° below the x -axis.