Problem 1 - The scalar transport equation. (Total 15 pts) (a) (5 pts) Start with the form of the conserved scalar transport equation given in Lecture 1:

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (\zeta \mathbf{u}) - D_{AB} \nabla^2 \zeta = 0.$$  \hfill (1)

Simplify this equation for the case where the fluid is incompressible, so that the condition $\nabla \cdot \mathbf{u} = 0$ holds. Do not rely on vector identities – work out the relevant derivatives explicitly.

(b) (10 pts) In studies of turbulent scalar mixing, the scalar ‘energy’, defined (analogously to kinetic energy) as $\frac{1}{2} \zeta^2$, often appears. Starting with the incompressible form of the conserved scalar transport equation derived in part (a), derive the transport equation for the scalar energy, i.e. the equation of the form

$$\frac{\partial \left( \frac{1}{2} \zeta^2 \right)}{\partial t} = \ldots.$$  

Explain what each of the terms in the transport equation means physically. Is scalar energy a conserved quantity?

Problem 2 - 2-D dynamic systems. (20 pts) Consider the following two-dimensional linear system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$  \hfill (2)

Without solving the system explicitly, explain if the system is stable or unstable near the origin. Draw some approximate trajectories.

Now consider the following non-linear system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \sin x_1 \\ 0 \end{pmatrix}.$$  \hfill (3)

Again, without solving the system, is this non-linear system stable or unstable near the origin? Draw some approximate trajectories if necessary.

Problem 3 - Planar turbulent jets. (15 pts) A planar jet is the flow that results when fluid issues from an infinitely long linear source. Suppose that the linear source is aligned with the $z$-axis, and that the flow issues in the $x$-direction. The planar jet is defined by $J_0$, its initial momentum flux per unit span (where $z$ is the spanwise direction). Suppose that the jet is turbulent. Assuming that the flow is self-similar, where the velocity profiles have width $\delta(x)$ and where the centerline velocity is $u_c(x)$, determine the dependence of $\delta$ and $u_c$ on $x$ (explain any assumptions you make). Compare the results with the axisymmetric jet case discussed in class.